STUDIES IN INTEGRABILITY OF NONLINEAR DYNAMICAL SYSTEMS

SYNOPSIS

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Introduction

Dynamical systems can be classified into linear dynamical systems and nonlinear dynamical systems. In mathematics, a nonlinear system is a system which is not linear, that is, a system which does not satisfy the superposition principle, or whose output is not directly proportional to its input. Nonlinear problems are of interest to engineers, physicists and mathematicians because most physical systems are inherently nonlinear in nature. In the nonlinear paradigm, an initial change in one variable does not produce a proportional change in the resultant variable. Nonlinear systems are modeled by nonlinear differential equations, ordinary or partial differential equations (ODEs & PDEs). Nonlinear equations are notoriously difficult to solve, but give rise to interesting phenomena such as chaos and solitons. Chaotic systems which are associated with nonintegrability, exhibit completely irregular and unpredictable behavior.Contrary to this solitonic systems also known as integrable systems, are characterized by regular or predictable behavior for all times of evolution.

A remarkable development in our understanding of a certain classes of nonlinear PDEs known as evolution equations is the fact that they possess a special type of elementary solution in the form of localized disturbances or pulses that retain their shape even after interaction among themselves. Thus they act like particles. These localized disturbances have come to be known as a solitons. The existence of solitons in nonlinear dynamical systems is closely related to their integrability. Integrability is defined as a property that guarantees the existence of complete set of invariants or conserved quantities associated with a system described by ordinary or partial differential equations. This set is finite for systems modeled by ordinary differential equations, but finite for partial differential equations. From a physical point of view, the existence of such integrable nonlinear dynamical systems often means the existence of very regular motion and from a mathematical

point of view they imply the existence of beautiful analytic and geometric structures. But an arbitrarily small change in the integrable equation can destroy its integrability.

After interaction a solitary waves emerge unaffected in their amplitude and velocities, except for phase shifts. The name 'Soliton' was coined by Zabuski-Kruskal in analogy with particles. The new concepts that emerge from Zabuski-Kruskal experiments are

1) A solitary wave can arise when the nonlinearity balances linear dispersion.

2) In appropriate nonlinear systems, these solitary waves can interact elastically like particles without changing their shape and velocities.

The development of mathematical properties of a large class of solvable nonlinear evolution equation started with the first recorded observation of the 'great solitary wave' by Scott Russell in 1834. The Dutch Physicist Korteweg and de Vries in 1895 deduced a famous wave equation based on Scott Russell's explanation and obtained the "Korteweg-de Vries" (KdV) equation given by

$$U_t + 6UU_{xx} + U_{xxx} = 0 \tag{1}$$

They had observed that the delicate balance between the nonlinear term (UU_x) and dispersion term (U_{xxx}) in the KdV equation results in a solitary wave pulse which moves with uniform velocity proportional to the amplitude.

The other classes of nonlinear evolution equations studied in soliton theory include the sine-Gordan (sG) and the nonlinear Schrödinger equations (NLSE). It is an integrable equation which can be solved with the Inverse Scattering Transform (IST) discovered by Gardener et al.

The Nonlinear Cubic Schrödinger Equations (NLCSE) is given by

$$iq_t + q_{xx} + |q|^2 q = 0 \tag{2}$$

The present study has been conceived with an idea to study the integrability properties of higher order NLSEs, to solve the coupled higher order NLSE (modified Manakov Model) using the

Hirota Bilinearization Technique and to study wave propagation in photorefractive media using Manakov Model.

Objectives

- To study the integrability properties of higher order NLSEs
- To find the soliton solutions for the coupled NLSE (modified Manakov Model)
- To study the application of Manakov Model in photorefractive media

Methods Used

- Integrability is studied using the Lax Method & the Painleve Analysis
- Soliton solutions are determined using the Hirota Bilinearization technique
- The existence of soliton in photorefractive media is studied using two beam coupling.

In Chapter 1, the history of study of Solitons and application of solitons in nonlinear optics are discussed. In optics, the NLSE occurs in the Manakov system, a model of wave propagation in fiber optics. Chapter 2 mainly concentrates on the aspect of integrability, integrable systems , the techniques for the determination of integrability mainly, the Lax Method and the Painleve analysis and the Hirota bilinear method of solving integrable equations.

In 1968, Lax put forward the Lax Method to find integrable equations.

Given the two differential operators

$$L = \frac{\partial^2}{\partial x^2} + U \tag{3}$$

and

$$M = \frac{\partial^{2n+1}}{\partial x^{2n+1}} + \text{lower degree} \tag{4}$$

Lax derives the equation

$$L_t = [M, L] \tag{5}$$

Lax proved that the initial data $U(x,0)$ determine the solution of such nonlinear evolution equations in the 'Lax pair' form $L_t = [M, L]$.This method remained the most powerful technique for developing analytic solutions of such equations.

The French mathematician Paul Painleve following the ideas of Fuchs, Kovalevskya and others, completely classified first order equations and second order equations. The main four steps involving in the P-analysis are:-

1. Determination of leading order behavior.
2. Determination of 'resonances', that is, the powers at which arbitrary constants of the solution can enter into the Laurent series expansion.
3. Verifying that at resonant values sufficient number of arbitrary functions exist without the introduction of movable critical points.
4. Establishing connections with the soliton and other integrability.

In 1971 Hirota introduced a new direct method for constructing multisoliton to integrable nonlinear evolution equations. The idea was to make transformation into new variables, so that, in these new variables multisoliton solutions appear in a particularly simple form. The method turned out to be very effective and was shown to give N-soliton solutions to KdV, modified KdV, sine - Gordan (sG) and NLSE.

The Hirota bilinear formalism has played a crucial role in the study of integrable system. In this method, a dependent variable is replaced by a fractional form, say $\frac{g(x,t)}{f(x,t)}$ and the equation to be satisfied by $g(x,t)$ and $f(x,t)$ is obtained in bilinear form.

In Chapter 3, the integrability properties of higher order NLSEs mainly the Nonlinear Cubic-Quintic Schrodinger Equation (NLCQSE) and the Nonlinear Septic Schrodinger equation (NLSSE) is studied using the Painleve Analysis and the Lax Method. Chapter 4 introduces the modified Manakov system in which cubic and quintic interactions are present. The applications of the Manakov model in photorefractive media is studied in Chapter 5.The coupled beam propagation through photorefractive media is investigated using the theory of two beam coupling. The process of two beam coupling has been studied theoretically and experimentally in many photorefractive materials. This

equation is solved analytically to find the conditions under which soliton like waves can propagate.

Result & Discussions

From the integrability studies it is found that NLCQSE admits required number of arbitrary functions without the introduction of movable singularity manifolds. Since the dominant terms are half integer, NLCQSE is partially integrable in Painleve sense. It is also found that the NLCQSE is partially integrable in Lax sense as well. Applying the Hirota bilinear method to the modified Manakov Model ,the one and two soliton solutions are obtained. Since the existence of soliton solution is a pointer towards integrability, it is believed that the modified system is integrable. However when the Hirota method is applied to quintic Manakov model, no soliton solution were obtained, indicating that the system is most probably nonintegrable. Applying the theory of two beam coupling the nonlinear wave propagation through photorefractive media using Manakov model and obtained the CNLSE. This equation is solved to analytically to find the conditions under which soliton like waves can propagate. It is observed that with an appropriate choice of the experimental parameters, the input beam profile can be made to converge asymptotically to a soliton state whose PR nonlinearity compensates for the diffraction.